Solutions to HW#6

1. (a) $x=0$ is the only point at which f is continuous; If $y \in (-\epsilon, \epsilon)$, $|f(y)-0| \leq |y| < \epsilon$. Thus $\lim_{y \to 0} f(y) = f(0)$.
If $x \neq 0$, there is a sequence of rationals $r_n \mapsto x$ and irrationals $p_n \mapsto x$. Clearly $f(r_n) = r_n \mapsto x$, while $f(p_n) = 0 \not\mapsto x$. Therefore the sequence

$f(r_1), f(p_1), f(r_2), f(p_2), \ldots$ does not converge even though the sequence $r_1, p_1, r_2, p_2, \ldots$ converges to x. Hence f is not continuous at x.

(b) f is continuous at a point x if and only if $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} f(p_n)$ where $\{r_n\}$ is any sequence of rational satisfying $r_n \mapsto x$ and $\{p_n\}$ is any sequence of irrationals satisfying $p_n \mapsto x$.
Notice that $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} r_n = x$, while $\lim_{n\to\infty} f(p_n) =$
$= \lim_{n\to\infty} (1-p_n) = 1-x$. Hence f is continuous at x if and only if $x = 1-x$ or when $x = \frac{1}{2}$.

(c) Clearly f is not continuous at any rational $x>0$: If $\{p_n\} \subset [0,1] \setminus \mathbb{Q}$ is any sequence satisfying $p_n \mapsto x$, then $f(p_n) = 0 \not\mapsto x$. If $x=0$ or $x \in [0,1]\setminus \mathbb{Q}$, then f is continuous at x: For any $\epsilon>0$ there is an integer N such that $\frac{1}{N} < \epsilon$.

Since $x \neq \frac{m}{n}$ for any $m, n \in \mathbb{N}$, there is some $\delta_k > 0$ such that the interval $(x-\delta_k, x+\delta_k)$ has no points of the form $\frac{m}{k+1}$. Let $\delta = \min\{\delta_1, \dots, \delta_{N-1}\}$. Then the interval $(x-\delta, x+\delta)$ contains no points of the form $\frac{m}{n}$ for $n = 2, 3, \dots, N$. Hence, if $y \in (x-\delta, x+\delta)$, $|f(y) - f(x)| = |f(y) - 0| \leq \frac{1}{n}$ for $n \geq N+1$ so $|f(y) - 0| < \epsilon$ which proves that f is continuous at x.

2. If $x \in A \cap B$, $\chi_A(x) + \chi_B(x) = 2$, whereas $\chi_{A\cup B}(x) \leq 1$. Hence the formula $\chi_{A\cup B} = \chi_A + \chi_B$ is not correct. Notice, however, that $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$ (why?)
The formula $\chi_{A\cap B} = \chi_A \cdot \chi_B$ is correct, because $x \in A\cap B$ if and only if $x \in A$ and $x \in B$. Thus $x \in A \cap B$ implies that $1 = \chi_{A\cap B}(x) = 1 \cdot 1 = \chi_A(x) \cdot \chi_B(x)$. If, on the other hand, $x \notin A \cap B$, we may assume without loss of generality that $x \notin A$. Hence $0 = \chi_{A\cap B}(x) = 0 \cdot \chi_B(x) = \chi_A(x) \cdot \chi_B(x)$.
Finally, the formula $\chi_{A\setminus B} = \chi_A - \chi_B$ is not correct, because if $x \notin A$ and $x \in B$, $\chi_{A\setminus B}(x) = 0$, while $\chi_A(x) - \chi_B(x) = -1$.
The correct formula is $\chi_{A\setminus B} = \chi_{A\cup B} - \chi_B$ (why?)

3. $\chi_C(f(x)) = 1$ if and only if $f(x) \in C$. Thus $\chi_C \circ f = \chi_{f(A)\cap C}$

4. Notice that $\chi_\Delta^{-1}(B_{1/3}(1)) = \Delta$. Since Δ is nowhere dense, we see that $\mathrm{int}(\Delta) = \emptyset$, which means that Δ contains no open intervals. Thus, χ_Δ is not continuous at any $x \in \Delta$ (Otherwise we would have $\chi_\Delta(B_\delta(x)) \subset B_{1/3}(1)$ for some $\delta > 0$). Similarly, since $\chi_\Delta^{-1}(B_{1/3}(0)) = \mathbb{R}\setminus\Delta$ and $\mathbb{R}\setminus\Delta$ is open, we see that χ_Δ is continuous at

each $x \in \mathbb{R} \setminus \Delta$ (why?)

5. Suppose $A \subset \mathbb{R}$ and $x \in \text{int}(A)$. Then, for some $\delta > 0$, $B_\delta(x) = (x-\delta, x+\delta) \subset \text{int}(A) \subset A$ and $\{1\} = \chi_A(B_\delta(x)) \subset$ $\subset B_\epsilon(1) = B_\epsilon(\chi_A(x))$ for any $\epsilon > 0$. Hence χ_A is continuous at each $x \in \text{int}(A)$.

Similarly, if $x \in \text{int}(A^c)$, then for some $\delta > 0$, $B_\delta(x) \subset \text{int}(A^c) \subset$ $\subset A^c$. Hence $\{0\} = \chi_A(B_\delta(x)) \subset B_\epsilon(0) = B_\epsilon(\chi_A(x))$ for any $\epsilon > 0$, implying that χ_A is continuous at each $x \in \text{int}(A^c)$.

Lastly, if x is on the boundary of A, that is, if $x \notin \text{int}(A)$ and $x \in \text{int}(A^c)$, then x is a limit point of both A and A^c. Therefore χ_A is discontinuous at x.

6. Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous. Then for $x \in \{x : f(x) > 0\}$, $f(x) = r > 0$. To show that $\{x : f(x) > 0\}$ is open, consider the open ball $B_{r/4}(f(x)) = (f(x) - \frac{r}{4}, f(x) + \frac{r}{4}) = (\frac{3r}{4}, \frac{5r}{4})$. Because f is continuous, the inverse image of any open set is open.

In particular, $f^{-1}(B_{r/4}(f(x)))$ is open and contains x. Therefore $B_\delta(x) \subset f^{-1}(B_{r/4}(f(x)))$ for some $\delta > 0$. Notice however that this implies that $f(B_\delta(x)) \subset B_{r/4}(f(x)) = (\frac{3r}{4}, \frac{5r}{4})$. Hence if $y \in B_\delta(x)$, $f(y) > \frac{3r}{4} > 0$ and we see that $B_\delta(x) \subset \{x : f(x) > 0\}$ which proves that this set is open.

Notice also that the set $\{x : f(x) < 0\}$ is open since it is identical to the set $\{x : -f(x) > 0\}$.

Thus the set $\{x : f(x) \neq 0\} = \{x : f(x) \neq 0\} = \{x : f(x) > 0\} \cup$ $\cup \{x : f(x) < 0\}$ is open as well, implying that $\{x : f(x) = 0\} =$ $= \{x : f(x) \neq 0\}^c$ is closed.

7. (a) Suppose $f : M \mapsto \mathbb{R}$ is continuous. Fix $a \in \mathbb{R}$. Then $g : M \mapsto \mathbb{R}$ given by $g(x) = f(x) - a$ is also continuous. Observe that the set $\{x : g(x) > 0\} = \{x : f(x) > a\}$ and $\{x : g(x) < 0\} =$ $= \{x : f(x) < a\}$. Now repeat the argument from problem 6 to prove that $\{x : g(x) > 0\}$ and $\{x : g(x) < 0\}$ are open.

(b) Suppose that the sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open for every $a \in \mathbb{R}$. Fix $y \in M$ and $\epsilon > 0$ and consider $f^{-1}(f(y) - \epsilon, f(y) + \epsilon) = \{x : f(y) - \epsilon < f(x) < f(y) + \epsilon\} =$ $= \{x : f(x) > f(y) - \epsilon\} \cap \{x : f(x) < f(y) + \epsilon\}$. Observe that $f^{-1}(B_\epsilon(f(y))) = f^{-1}(f(y) - \epsilon, f(y) + \epsilon)$ is the intersection of two open sets and must therefore be open. Let $\delta > 0$ satisfy $B_\delta(y) \subset f^{-1}(B_\epsilon(f(y)))$ [Clearly, $y \in \{x : f(y) - \epsilon < f(x) < f(y) + \epsilon\}$]. This proves that f is continuous at y. Since y is arbitrary, it follows that f is continuous.

(c) for $y \in M$ and $\epsilon > 0$, fix rational numbers α and β satisfying $f(y) - \epsilon < \alpha < f(y) < \beta < f(y) + \epsilon$. Then $f(y) \in (\alpha, \beta) \subset$ $\subset (f(y) - \epsilon, f(y) + \epsilon)$. Hence $f^{-1}(\alpha, \beta) = \{x : \alpha < f(x) < \beta\} =$ $= \{x : f(x) > \alpha\} \cap \{x : f(x) < \beta\}$ is open. Furthermore, $y \in f^{-1}(\alpha, \beta) \subset$ $\subset f^{-1}(f(y) - \epsilon, f(y) + \epsilon) = f^{-1}(B_\epsilon(f(y)))$ so $B_\delta(y) \subset f^{-1}(\alpha, \beta) \subset f^{-1}(B_\epsilon(f(y)))$

This proves that f is continuous at y. Again, since y is arbitrary, we have shown that f is continuous.

8. (a) Let $f(0) = r > 0$. Then $B_{r/2}(f(0)) = (\frac{1}{2}r, \frac{3}{2}r)$. Since f is continuous at 0, there is some $a > 0$ that satisfies $f(B_a(0)) = f(-a, a) \subset B_{r/2}(f(0)) = (\frac{1}{2}r, \frac{3}{2}r)$. Hence $f(x) > \frac{1}{2}r > 0$ for all $x \in (-a, a)$

(b) If there were an x for which $f(x) < 0$, by a slight modification of the argument in part (a), there would be some interval $(x-a, x+a)$ on which f is negative. However, since the interval $(x-a, x+a)$ contains rationals and f is nonnegative on every rational, such interval cannot exist. Thus in particular $f(x) \geq 0$ for all x

9. Let $A = (0,1] \cup \{2\}$ have the usual metric function of $\mathbb{R}$. Then the open ball $B^A_{1/2}(2) = (2-\frac{1}{2}, 2+\frac{1}{2}) \cap A = \{2\}$ is an open subset of A. If $f: A \to \mathbb{R}$ is any function and $\epsilon > 0$, $f(B^A_{1/2}(2)) = \{f(2)\} \subset B_\epsilon(f(2))$. This shows that any function $f: A \to \mathbb{R}$ is continuous at 2.

10. (a) If f is continuous at each point of A (relative to M) and each point of B (relative to M), then f is continuous at each point of $A \cup B$ (relative to M). This is so because for each $x \in A \cup B$, we may assume without loss of generality that $x \in A$. Therefore by hypothesis x is a point of continuity of f relative to M.

(b) Suppose $f|_B$, $f: M \mapsto \mathbb{R}$ restricted to B, is continuous relative to B and $f|_A$, $f: M \mapsto \mathbb{R}$ restricted to A, is continuous relative to A. Then it is not necessarily true that $f|_{A \cup B}$ is continuous relative to $A \cup B$. To see this, consider $\chi_{\mathbb{Q}}: \mathbb{R} \to \mathbb{R}$ and set $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$. Then $\chi_{\mathbb{Q}}|_A = 1$, $\chi_{\mathbb{Q}}|_B = 0$ are constant functions. Therefore $\chi_{\mathbb{Q}}|_A$ is continuous relative to A and $\chi_{\mathbb{Q}}|_B$ is continuous relative to B. However $\chi_{\mathbb{Q}}|_{A \cup B} = \chi_{\mathbb{Q}}$ is not continuous anywhere on $A \cup B$.
Notice that $f|_{A \cup B}$ is continuous whenever $f|_A$ and $f|_B$ are continuous with the added hypothesis that A and B are open subsets of M. (Why?)

11. First, partition the interval $[0, 1]$ into interval subsegments using the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$

For each integer $n \in \mathbb{N}$ set $I_n = (\frac{1}{n+1}, \frac{1}{n}) \cap I$, where $I = (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$.
Then the I_n are pairwise disjoint, nonempty open subsets of (relative to) I.
Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers $\mathbb{Q} \cap [0,1]$
Define $g: (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \mapsto \mathbb{Q} \cap [0,1]$ by $g(x) = q_n$ if $x \in I_n$.
Clearly g is onto. To see that g is continuous, observe that $g^{-1}(\{q_n\}) = I_n$. That is, the inverse image under g of any singleton is an open set. From this it immediately follows that for any $V \subset \mathbb{Q} \cap [0,1]$, $g^{-1}(V)$ is an open subset of I. Thus the inverse image of any open

set is open and therefore g is continuous.

12. Define $k: M \mapsto \mathbb{R}$ by $k(x) = \rho(f(x), g(x))$. We will prove that $k(x) = 0$ for all $x \in M$, showing that $f(x) = g(x)$.
Notice that $|k(x) - k(y)| = |\rho(f(x), g(x)) - \rho(f(y), g(y))| \leq$
$\leq |\rho(f(x), g(x)) - \rho(f(x), g(y))| + |\rho(f(x), g(y)) - \rho(f(y), g(y))| \leq$
$\leq \rho(g(x), g(y)) + \rho(f(x), f(y))$

Since $f, g: (M, d) \mapsto (N, \rho)$ are continuous, the calculation above implies that $k: M \mapsto \mathbb{R}$ is continuous. (Why?)
Since $f(x) = g(x)$ for any $x \in D$, we have $k(x) = 0$ for all $x \in D$.
If, for some $y \in M$, $k(y) \neq 0$, we may assume without loss of generality that $k(y) > 0$. By a slight modification of problem 8(a), there is a neighborhood $B_\delta^d(y) \subset M$ such that $k(z) > 0$ for any $z \in B_\delta^d(y)$.
But $D \cap B_\delta^d(y) \neq \emptyset$, since D is dense in M. Let $w \in D \cap B_\delta^d(y)$.
Then $k(w) = 0$, contradicting the assertion that k is positive on $B_\delta^d(y)$. We conclude that $k(x) = 0$ for all $x \in M$, from which the desired result follows.

Suppose that $f: (M, d) \mapsto (N, \rho)$ is onto. Then any $z \in N$ is of the form $f(x)$ for some $x \in M$. Since f is continuous at x, for any $\epsilon > 0$ there is a $\delta > 0$ such that $f(B_\delta^d(x)) \subset B_\epsilon^\rho(f(x))$. Since $D \cap B_\delta^d(x)$ is not empty, $f(D) \cap B_\epsilon^\rho(f(x))$ is also not empty. In particular, either $f(x) \in f(D)$ or $f(x)$ is a limit point of $f(D)$. This establishes that $f(D)$ is dense in N.

13. Let $f(x) = \sin x$. Recall from calculus that f is everywhere differentiable. Thus, for any fixed $x, y \in \mathbb{R}$, f is continuous on

$[x,y]$ and differentiable on (x,y). Therefore, by the mean-value theorem, there is some $c \in (x,y)$ such that $\frac{f(x)-f(y)}{x-y} = f'(c)$. This means that $\sin x - \sin y = \cos c\,(x-y)$ so $|\sin x - \sin y| = |\cos c|\,|x-y| \leq |x-y|$. In particular, $f(x) = \sin x$ is Lipschitz with constant $K = 1$.

Observe that every Lipschitz function of order K is continuous; $|f(x)-f(y)| < \epsilon$ whenever $|x-y| < \frac{\epsilon}{K}$.

14. If $f:(M,d) \mapsto (N,\rho)$ is Lipschitz, then $\rho(f(x),f(y)) \leq K d(x,y)$ means that $\rho(f(x),f(y)) < \epsilon$ whenever $d(x,y) < \frac{\epsilon}{K}$. In particular, f is continuous.

15. Let $f,g \in C[a,b]$. Define $L: C[a,b] \mapsto \mathbb{R}$ by $L(f) = \int_a^b f(t)\,dt$. Then $|L(f)-L(g)| = \left|\int_a^b (f(t)-g(t))\,dt\right| \leq \int_a^b |f(t)-g(t)|\,dt \leq$
$\leq \int_a^b \|f-g\|_\infty\,dt = |b-a|\,\|f-g\|_\infty$. Setting $K = |b-a|$, we see that L is Lipschitz with constant K.

16. Define $g: l_2 \mapsto \mathbb{R}$ by $g(x) = \sum_{n=1}^{\infty} \frac{x_n}{n}$. Then $|g(x)-g(y)| =$
$$= \left|\sum_{n=1}^{\infty} \frac{x_n - y_n}{n}\right| \leq \sum_{n=1}^{\infty} \frac{|x_n-y_n|}{n} \leq \left(\sum_{n=1}^{\infty} |x_n-y_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2}$$
$= K\|x-y\|_2$. Therefore g is Lipschitz.